TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 190, 1974

SINGULAR PERTURBATIONS FOR SYSTEMS OF LINEAR PARTIAL DIFFERENTIAL EQUATIONS

BY

A. LIVNE AND Z. SCHUSS

ABSTRACT. We consider the system of linear partial differential equations $\epsilon A^{ij}u^\epsilon_{ij} + B^i u^\epsilon_i + Cu^\epsilon = f$ where A^{ij}, B^i are symmetric $m \times m$ matrices and $-C$ is a sufficiently large positive definite matrix. We prove that under suitable conditions $\|u^\epsilon - u\|_{L^2} \le c\sqrt{\epsilon}\|f\|_{H^1}$ where u is the solution of a suitable boundary value problem for the system $B^i u_i + Cu = f$.

1. **Introduction.** This paper is concerned with the linear differential operator

$$L_\epsilon u = \epsilon A^{ij}u_{ij} + B^i u_i + Cu = f \tag{1.1}$$

for $0 < \epsilon < \epsilon_0$, where A^{ij}, B^i, C are $m \times m$ matrix valued functions defined on a compact domain $\mathfrak{M}$ in R^n, whose boundary $\dot{\mathfrak{M}}$ is of class C^∞. The operator acts on vector-valued functions likewise defined on $\mathfrak{M}$ and subject to homogeneous boundary conditions. We assume A^{ij}, B^i, C are symmetric or symmetrizable in the sense of [10]. We wish to compare the solution u_ϵ of (1.1) with the solution v of the reduced problem

$$B^i v_i + Cv = f. \tag{1.2}$$

Our purpose is to find the rate of convergence of u_ϵ to v for $\epsilon \to 0$ i.e. to derive bounds of the form $\|u_\epsilon - v\| \le C\sqrt{\epsilon}\|f\|_1$, where $\|\cdot\|$, $\|\cdot\|_1$ are suitable norms. Recently S. V. Sivašinskiĭ [14], C. Bardos, D. and H. Brezis [2] considered the question of strong convergence of u_ϵ to v for similar operators. The problem of deriving bounds on the rate of convergence of singular perturbations for elliptic and parabolic equations was treated by Greenlee [5], Friedman [3], the authors [12] and others. In [5] and [3] a single elliptic or parabolic equation of higher order degenerates into a lower order elliptic or parabolic equation and the rate of convergence of u_ϵ to v is obtained. In [12] we considered the case of a single second order equation with nonnegative characteristic form that degenerates into

Received by the editors November 16, 1972 and, in revised form, May 13, 1973.
AMS (MOS) subject classifications (1970). Primary 35F15, 35B25; Secondary 35B25, 35L40, 35B30.
Key words and phrases. First order system, singular perturbations, rate of convergence.

a first order equation. The methods developed in [12] are used here for systems of equations. The asymptotic nature of the solution as $\epsilon \to 0$ was thoroughly investigated by Vishik and Liusternik [15], Oleinik [13] and others. Equation (1.1) is frequently used as an "elliptic regularization" of (1.2), e.g. by Kohn and Nirenberg [7], [8] and many other authors. They prove weak convergence in a suitable space of some sequence u_{ϵ_n} to a solution v of (1.2). However, strong convergence and its rate are not discussed. The problem of determining the boundary conditions satisfied by the limit function v was discussed by Levinson [11], Ladyzhenskaya [9], Kamenomostskaya [6], Sivašinskiĭ [14], Bardos and Brezis [2], the authors [12] and others. It is worthwhile to note that no maximum principle holds for the problem under consideration, unlike the situation in [6], [8], [9], [11], [12], [13]. We wish to thank A. Friedman and H. Brezis for bringing the problem to our attention, and to C. Bardos for considerably simplifying the proof of our last theorem.

2. **Preliminary results and notation.** Let $\mathfrak{M}$ be a compact domain in R^n as above and let $x = (x^1, \cdots, x^n)$ represent the coordinates. We use subscripts to denote differentiation, and also use the summation convention. The coefficients are real and of class C^∞ in $\overline{\mathfrak{M}}$, the closure of $\mathfrak{M}$. We use ν^j to denote the jth component of the unit exterior normal at $\dot{\mathfrak{M}}$. The boundary matrix $B_\nu(x)$ is defined by

$$B_\nu = \sum_{j=1}^{n} B^j \nu^j.$$

We make the following assumptions.

(a) *There exists a positive constant* α *such that for any* $\xi \in R^n$, $|\xi| = 1$, *and any* $\eta \in R^m$

$$(A^{ij}\xi^i\xi^j\eta, \eta) \geq \alpha|\eta|^2. \tag{2.1}$$

Here (,) denotes the scalar product in the relevant Euclidean space, $|\cdot|$ denotes the Euclidean norm.

We assume that the boundary $\dot{\mathfrak{M}}$ consists of two disjoint parts Σ_1 and Σ_2 such that

(b) *The boundary matrix* B_ν *is negative definite on* Σ_1 *and positive definite on* Σ_2, *i.e. there exists a positive constant* β *such that for any* $\eta \in R^m$

$$-(B_\nu\eta, \eta) \geq \beta|\eta|^2 \quad on\ \Sigma_1, \tag{2.2}$$

$$(B_\nu\eta, \eta) \geq \beta|\eta|^2 \quad on\ \Sigma_2. \tag{2.2'}$$

(c) *The matrix* $-C$ *is sufficiently large positive definite, i.e. there exists a sufficiently large positive constant* γ *(independent of* ϵ*) such that, for any* $\eta \in R^m$, $-(C\eta, \eta) \geq 2\gamma|\eta|^2$. We use

$$\langle u, v\rangle = \int_{\mathfrak{M}} (u, v)\,dV, \qquad \langle u, v\rangle_{\dot{\mathfrak{M}}} = \int_{\dot{\mathfrak{M}}} (u, v)\,ds, \qquad \|u\|^2 = \langle u, u\rangle,$$

where dV and ds are the volume elements on $\mathfrak{M}$ and $\dot{\mathfrak{M}}$ respectively. Similarly, $\|u\|_k^\Omega$ denotes the norm of $u(x)$ in $(H^k(\Omega))^m$ for any subdomain $\Omega \subset \mathfrak{M}$ (cf. [1]).

If conditions (a), (b), (c) are satisfied then it can be shown (cf. [1, Lemma 2.1]) that there exists ϵ_0 such that for all $\epsilon_0 > \epsilon > 0$ $\langle -L_\epsilon u, u\rangle \geq \gamma\|u\|^2$ for all $u \in (C^\infty(\bar{\mathfrak{M}}))^m$ satisfying $u = 0$ on Σ_2, and $\partial u/\partial \nu = 0$ on Σ_1. It follows [7] that for any $f \in (L_2(\mathfrak{M}))^m$ the mixed boundary value problem

$$\begin{aligned} L_\epsilon u &= f \quad \text{in } \mathfrak{M} \quad (\epsilon > 0), \\ \partial u/\partial \nu &= 0 \quad \text{on } \Sigma_1, \qquad (2.3) \\ u &= 0 \quad \text{on } \Sigma_2, \end{aligned}$$

has a unique solution $u \in (H^2(\mathfrak{M}))^m$, satisfying the boundary conditions in the strong sense [1] and $\gamma\|u\| \leq \|f\|$. If $f \in (H^k(\mathfrak{M}))^m$ then $u \in (H^{k+2}(\mathfrak{M}))^m$ $(k \geq 1)$. Following [8] we introduce local coordinates near $\dot{\mathfrak{M}}$: Let $-y$ denote the distance from the boundary; on the surfaces $y = \text{const}$, we use $x' = (x^1, \cdots, x^{n-1})$ as local coordinates, taking $x^n = y$ after a change of coordinates. In such coordinates (x', y) we denote the leading coefficients of L_ϵ by ϵA^{ij} for $i, j = 1, \cdots, n-1$; otherwise we set $A^{nn} = A$, $2A^{ni} = A^i$ for $i \neq n$, and $B^n = B$, so that L_ϵ takes the form

$$L_\epsilon u = \epsilon A u_{yy} + B u_y + \epsilon A^i u_{iy} + \epsilon A^{ij} u_{ij} + B^i u_i + Cu. \tag{2.4}$$

Here summation over i, j extends from 1 to $n-1$; the subscripts i, j, y denote differentiation with respect to x^i, x^j, $i, j < n$, and y. To be more precise, let $x_0 \in \dot{\mathfrak{M}}$ and let $y = \phi(x)$ be the distance of a point x, in a neighborhood of x_0, to $\dot{\mathfrak{M}}$ so that $\phi(x) < 0$ in $\mathfrak{M}$, and $\phi(x) > 0$ outside $\mathfrak{M}$. We have $\phi = 0$ on $\dot{\mathfrak{M}}$ and $\nabla\phi(x_0) \neq 0$; setting $\xi^i = x^i$, $i < n$, $\xi^n = y = \phi(x)$ we have, locally, the following formulas for the coefficients of L_ϵ:

$$\begin{aligned} A &= A^{ij}\phi^i\phi^j && (i, j = 1, \ldots, n), \\ A^i &= 2A^{ij}\phi_j && (j = 1, \cdots, n;\ i = 1, \cdots, n-1), \\ B &= \epsilon A^{ij}\phi_{ij} + B^i\phi_i && (i, j = 1, \cdots, n). \end{aligned}$$

Note that B is positive definite near Σ_2 and negative definite near Σ_1 if ϵ is sufficiently small.

3. **A priori estimates.**

Lemma 3.1. *Assume* (a), (b), (c) *hold; let* $f \in (H^1(\mathfrak{M}))^m$, *and* $u = u_\epsilon(x)$ *be the solution of* (2.3). *Then there exist neighborhoods* $\mathfrak{M}_2$ *and* $\mathfrak{A}_2$ *of* Σ_2 (*in* $\bar{\mathfrak{M}}$) *such that* $\bar{\mathfrak{M}}_2 \subset \mathfrak{A}_2$, *and constants c* (*independent of* ϵ, y) *and K* (*independent of* ϵ) *such that*

$$y^{1/2}\|u\|_1^{\mathfrak{M}_2 \, 2} \le c\|u\|_1^{\mathfrak{A}_2 \, 2} + K\|f\|_1^{\mathfrak{A}_2 \, 2}.$$

Proof. The boundary Σ_2 may be covered by a finite number, say k, of small neighborhoods U_j, each of which may be described in terms of our local coordinates by

$$|x^i - x_0^i| < d \quad (i = 1, \cdots, n-1), \quad 0 \le -y < d,$$

for a small constant d. These neighborhoods may be chosen in such a way that any one of the enlarged regions

$$U_j' : |x^i - x_0^i| < 2d \quad (i = 1, \cdots, n-1), \quad 0 \le -y < 2d,$$

intersects at most 3^{n-1} of the other U_i'. We may assume that, for d suitably small, the changes from coordinates on U_j' to coordinates on any one of the other U_i' have Jacobians $\ge 1/2$ on $U_j' \cap U_i'$. We define

$$\mathfrak{M}_2 = \bigcup_{j=1}^{k} U_j, \quad \mathfrak{A}_2 = \bigcup_{j=1}^{k} U_j'.$$

Using local coordinates in each U_j, we define (cf. [8])

$$\kappa = \max_j \int_{U_j} \sum_{i=1}^{n-1} |u_i|^2 \, dV. \tag{3.1}$$

All terms which are bounded by $c\|u\|_1^{\mathfrak{A}_2 - \mathfrak{M}_2 \, 2} + K\|f\|_1^{\mathfrak{A}_2 \, 2}$ will be denoted by $H^{1/2}$, where c denotes any constant independent of ϵ, y and K is any constant independent of ϵ. Our aim is to show that

$$\kappa \le H. \tag{3.2}$$

Let U, U' be the neighborhoods where the maximum in (3.1) is achieved. We start by showing that $(\|u_y\|^{\mathfrak{M}_2 \, 2})^2 \le c\kappa + H$. Let ζ be a nonnegative function in $C^\infty(\bar{\mathfrak{M}})$ such that (i) $\zeta = 1$ in U, (ii) ζ is independent of y in $U' \cap \{0 \le -y < d\}$, (iii) $\zeta = 0$ outside U', and (iv)

$$|\zeta_i| \le c/d. \tag{3.3}$$

We take the scalar product of (2.4) with $\zeta^2 u_y$ to obtain

$$\epsilon\langle Au_{yy}, \zeta^2 u_y\rangle + \langle B\zeta u_y, \zeta u_y\rangle = -\sum_{j=0}^{4} I_j. \tag{3.4}$$

Clearly

$$|I_0| = |\langle f, \zeta^2 u_y\rangle| \leq H + \mu\|\zeta u_y\|^2, \qquad |I_1| = |\langle Cu, \zeta^2 u_y\rangle| \leq H \tag{3.5}$$

where μ is a small constant.

We have used here the estimate $\gamma\|u\| \leq \|f\|$ and integration by parts. Now let U_i', U_j' be any two neighborhoods such that $U_i' \cap U_j' \neq \emptyset$. Next, let $(x^1, \cdots, x^{n-1})$ and $(\xi^1, \cdots, \xi^{n-1})$ be local tangential coordinates in U_i' and U_j' respectively. If P is any point in $U_i' \cap U_j'$ then

$$\frac{\partial u(P)}{\partial \xi^j} = \sum_{i=1}^{n-1} J^{ij}(P)\frac{\partial u(P)}{\partial x^i} + J^{yj}\frac{\partial u(P)}{\partial y}$$

where $J(P)$ is the Jacobian matrix at P. We can choose d sufficiently small so that $J^{yj}(P)$ is small. Thus, for any $\mu' > 0$, d is chosen so small that

$$\int_{U_i'\cap U_j'} \sum_{j=1}^{n-1} \left|\frac{\partial u}{\partial x^j}\right|^2 dV_x \leq c(\mu') \int_{U_i'\cap U_j'} \sum_{j=1}^{n-1} \left|\frac{\partial u}{\partial \xi^j}\right|^2 dV_\xi + \mu' \int_{U_i'\cap U_j'} |u_y|^2\, dV. \tag{3.6}$$

Next we define $U'' = U' \cap (\mathfrak{A}_2 - U)$. Then, using (3.6), we find that for μ, μ' sufficiently small

$$|I_2| = |\langle B^i u_i, \zeta^2 u_y\rangle| \leq \mu\|\zeta u_y\|^2 + \mu'(\|\zeta u_y\|^{U''})^2 + c(\mu)\kappa.$$

$$I_3 = \epsilon\langle A^{ij}u_{ij}, \zeta^2 u_y\rangle \tag{3.7}$$

$$= -\epsilon\langle A^{ij}u_i, \zeta^2 u_{yj}\rangle - \epsilon\langle A_j^{ij}\zeta u_i, \zeta u_y\rangle - 2\epsilon\langle A^{ij}\zeta_j u_i, \zeta u_y\rangle = J_1 + J_2 + J_3.$$

J_2 may be estimated as in (3.5).

Next, we estimate J_1,

$$|J_1| = \tfrac{1}{2}\epsilon|\langle (A^{ij}\zeta^2)_y u_i, u_j\rangle| \leq c\kappa + H + \mu'(\|\zeta u_y\|^{U''})^2.$$

Next, to estimate J_3, we use (3.3) and get

$$|J_3| \leq \mu\|\zeta u_y\|^2 + (\epsilon^2 c(\mu)/d^2 + \mu')((\|u_y\|^{U''})^2 + c(\mu)\kappa + H).$$

Finally

$$(3.8)\qquad |I_4| = \epsilon|\langle A^i u_{iy}, \zeta^2 u_y\rangle| = \tfrac{1}{2}\epsilon|\langle (A^i\zeta^2)_i u_y, u_y\rangle| \le (c\epsilon/d)\|\zeta^{1/2}u_y\|^2.$$

Regarding the left-hand side of (3.4) we have

$$(3.9)\qquad \epsilon\langle Au_{yy}, \zeta^2 u_y\rangle \ge \tfrac{1}{2}\epsilon\langle A\zeta u_y, \zeta u_y\rangle_{\dot{\mathfrak{M}}} - \epsilon(c\|\zeta u_y\|^2 + H)$$

where the first term on the right-hand side is nonnegative. Now, since B is positive definite in $\mathfrak{A}_2$ we have, upon using (3.4), (3.5), (3.8), and (3.9),

$$(3.10)\qquad (\|u_y\|^U)^2 \le (\mu' + \mu + \epsilon/d)c(\|u_y\|^{U''})^2 + c\kappa + H.$$

We can replace U by any U_j in (3.10); hence, summing over j, we get

$$(\|u_y\|^{\mathfrak{M}_2})^2 \le (\mu' + \mu + \epsilon/d)c(\|u_y\|^{\mathfrak{M}_2})^2 + c\kappa + H.$$

Now, if μ, μ', ϵ are sufficiently small (μ, μ' independent of ϵ) we get

$$(3.11)\qquad (\|u_y\|^{\mathfrak{M}_2})^2 \le c\kappa + H.$$

We shall proceed to show that

$$(3.12)\qquad \gamma\kappa \le c(\|u_y\|^{\mathfrak{M}_2})^2 + H.$$

Then, using (3.11) and (3.12) we get $\kappa \le H$, if γ is sufficientiy large (independent of ϵ); the assertion of the lemma follows. Keeping the same notations, consider $\langle L_\epsilon u, (\zeta^2 u_j)_j\rangle = \langle f, (\zeta^2 u_j)_j\rangle$. We have

$$(3.13)\qquad -\langle C\zeta u_j, \zeta u_j\rangle \ge 2\gamma\kappa.$$

Integration by parts and the nonnegativity of $-\langle L_\epsilon u, u\rangle$ disposes of terms containing four derivatives, e.g. $\langle A^{ij}\zeta u_{ki}, \zeta u_{kj}\rangle$. Regarding terms containing three derivatives, e.g. $I = \langle B^i u_i, (\zeta^2 u_j)_j\rangle$, we have

$$I = -\langle B^i_j \zeta u_i, \zeta u_j\rangle - \langle B^i \zeta u_{ij}, \zeta u_j\rangle = J_1 + J_2.$$

J_1 is estimated by (3.13), while $J_2 = \tfrac{1}{2}\langle B^i_i \zeta u_j, \zeta u_j\rangle + \langle B^i \zeta u_j, \zeta_i u_j\rangle$. The first summand is estimated by (3.13) and the second by $c\kappa + (\|u_y\|^{U''})^2$. Hence, using (3.13), we get (3.12).

Lemma 3.2. *If the conditions of Lemma 3.1 are satisfied then there exist neighborhoods $\mathfrak{M}_1$ and $\mathfrak{A}_1$ of Σ_1 (in $\mathfrak{M}$) such that $\mathfrak{M}_1 \subset \mathfrak{A}_1$ and such that*

$$(3.14)\qquad \|u\|_1^{\mathfrak{M}_1} \le c\|u\|_1^{\mathfrak{A}_1 - \mathfrak{M}_1} + K\|f\|_1^{\mathfrak{A}_1}.$$

Proof. Let U_j be a local coordinate neighborhood of a point in Σ_1 as above. Proceeding as in the proof of the previous lemma we get the estimates (3.10), (3.11), (3.13), (3.14). Next, we estimate J_1,

$$J_1 = -\langle \epsilon A^{ij}\zeta u_i, \zeta u_{y_j}\rangle = -\tfrac{1}{2}\langle \epsilon A^{ij}\zeta u_i, \zeta u_j\rangle_{\Sigma_1} + \tfrac{1}{2}\langle (\epsilon A^{ij}\zeta^2)_y u_i, u_j\rangle$$

$$\leq c\kappa + H + \mu'(\|\zeta u_y\|^{U''})^2$$

since the boundary term is nonpositive. Multiplying (3.11) by -1 and noting that for any $\eta \in R^m$, $-(B\eta, \eta) \geq \tfrac{1}{2}\beta|\eta|^2$ in $\mathfrak{A}_1$, we see that for ϵ, μ, μ' sufficiently small we get

$$(\|u_y\|^{\mathfrak{M}_1})^2 \leq c\kappa + H. \tag{3.15}$$

The analogue of (3.12),

$$\gamma\kappa \leq c(\|u_y\|^{\mathfrak{M}_1})^2 + H, \tag{3.16}$$

is obtained in a similar way.

Theorem 3.1. *Assume* (a), (b), (c) *hold. Let* $f \in (H^1(\mathfrak{M}))^m$ *and u be the solution of* (2.3). *Then there exists a constant c, independent of* ϵ, *such that*

$$\|u\|_1 \leq c\|f\|_1 \tag{3.17}$$

if γ *is sufficiently large* (*independent of* ϵ).

Proof. Set $\mathfrak{N}_i = \{x \in \mathfrak{M} \mid \mathrm{dist}(x, \Sigma_i) < d/2\}$ $(i = 1,2)$. Notice that $\mathfrak{N}_i \subset \mathfrak{M}_i$ $(i = 1, 2)$. We define

$$\Omega = \overline{\mathfrak{M}} - \mathfrak{M}_1 - \mathfrak{M}_2, \qquad \Omega' = \overline{\mathfrak{M}} - \mathfrak{N}_1 - \mathfrak{N}_2.$$

Let ζ be a smooth function defined on $\overline{\mathfrak{M}}$ such that $\zeta = 1$ on Ω, $\zeta = 0$ outside Ω'. In view of Lemmas 3.1, 3.2 it suffices to show that

$$\gamma^{1/2}\|u\|_1^{\Omega} \leq c\|u\|_1^{\Omega'-\Omega} + K\|f\|_1. \tag{3.18}$$

This estimate is established by arguments similar to the ones used in the proof of the previous lemmas. To be more specific consider

$$\langle L_\epsilon u, (\zeta^2 u_k)_k\rangle = \langle f, (\zeta^2 u_k)_k\rangle$$

(in the usual Cartesian coordinate system). Integration by parts gives an estimate of the left-hand side by the norms of $u_i\zeta_j$, but since $\zeta_i = 0$ in Ω we get (3.18).

4. Singular perturbations. Let u_0 be the solution of the first order system

$$(4.1)\qquad \begin{aligned} B^i u_i + Cu &= f \quad \text{in } \mathfrak{M}, \\ u &= 0 \quad \text{on } \Sigma_2. \end{aligned}$$

It is well known [7], [10] that such a solution exists, $u_0 \in (H^1(\mathfrak{M}))^m$ if $f \in (H^1(\mathfrak{M}))^m$ and $\|u_0\|_1 \leq c\|f\|_1$. The main result of this paper is

Theorem 4.1. *Assume conditions* (a), (b), (c) *hold. Let* $f \in (H^1(\mathfrak{M}))^m$, *then the solutions* $\{u_\epsilon\}$ *of* (2.3) *converge in* $(L_2(\mathfrak{M}))^m$ *to* u_0, *and*

$$(4.2)\qquad \|u_\epsilon - u_0\| \leq c\sqrt{\epsilon}\|f\|_1,$$

where c is a constant independent of ϵ.

Proof. Denote by $\mathbf{A}$ the operator $-A^{ij}D_{ij}$ in $(L_2(\mathfrak{M}))^m = H$ whose domain $D(\mathbf{A})$ is the closure of the set $\{v \in (C^2(\overline{\mathfrak{M}}))^m \mid v = 0 \text{ on } \Sigma_2,\ \partial v/\partial \nu = 0 \text{ on } \Sigma_1\}$ in $(H^2(\mathfrak{M}))^m$. Similarly, let $\mathbf{B}$ be the operator $-(B^iD_i + C)$ in $(L_2(\mathfrak{M}))^m$ whose domain $D(\mathbf{B})$ is the closure of the set $\{v \in (C^1(\overline{\mathfrak{M}}))^m \mid v = 0 \text{ on } \Sigma_2\}$ in $(H^1(\mathfrak{M}))^m$. Then [2] $\mathbf{A}u_0 \in (H^{-1}(\mathfrak{M}))^m$. Therefore

$$(4.3)\qquad \epsilon\mathbf{A}(u_\epsilon - u_0) + \mathbf{B}(u_\epsilon - u_0) = -\epsilon\mathbf{A}u_0.$$

Now we take the scalar product of (4.3) with $u_\epsilon - u_0$ in H'. Since the operator $\mathbf{B}$ is strictly positive we get

$$\|u_\epsilon - u\|^2 \leq \epsilon c\|u_0\|_1\|u_\epsilon - u_0\|_1 \leq \epsilon c\|f\|_1^2$$

by Theorem 3.1. Thus Theorem 4.1 is proved.

Remark. The boundary conditions in (4.1) are very restrictive. They follow from the restrictive conditions (2.2), (2.2′). It would be desirable to replace them by the usual conditions of [4]. The question of strong convergence of u_ϵ to u_0 was settled in [2] and [14] for a wide class of boundary conditions, but the rate of convergence is still unknown.

REFERENCES

1. S. Agmon, *Lectures on elliptic boundary value problems*, Van Nostrand Math. Studies, no. 2, Van Nostrand, Princeton, N. J., 1965. MR 31 #2504.

2. C. Bardos, D. Brezis and H. Brezis, *Perturbations singulières et prolongements maximaux d'opérateurs positifs*, Arch. Rational Mech. Anal. (to appear).

3. A. Friedman, *Singular perturbations for partial differential equations*, Arch. Rational Mech. Anal. 29 (1968), 289–303. MR 37 #1754.

4. K. O. Friedrichs and P. D. Lax, *Boundary value problems for first order operators*, Comm. Pure Appl. Math. 18 (1965), 355–388. MR 30 #5186.

5. W. M. Greenlee, *Rate of convergence in singular perturbations*, Ann. Inst. Fourier (Grenoble) 18 (1968), fasc. 2, 135–191. MR 39 #3133.

6. S. Kamenomostskaya, *The first boundary problem for equations of elliptic type with a small parameter with the highest derivatives*, Izv. Akad. Nauk SSSR Ser. Mat. 19 (1955), 345–360. (Russian) MR 17, 627.

7. J. J. Kohn and L. Nirenberg, *Non-coercive boundary value problems*, Comm. Pure Appl. Math. 18 (1965), 443–492. MR 31 #6041.

8. ———, *Degenerate elliptic-parabolic equations of second order*, Comm. Pure Appl. Math. 20 (1967), 797–872. MR 38 #2437.

9. O. A. Ladyženskaja, *On equations with small parameter in the higher derivatives in linear partial differential equations*, Vestnik Leningrad. Univ. Ser. Mat. Meh. Astr. 12 (1957), no. 7, 104–120. (Russian) MR 19, 656.

10. P. D. Lax and R. S. Phillips, *Local boundary conditions for dissipative symmetric linear differential operators*, Comm. Pure Appl. Math. 13 (1960), 427–455. MR 22 #9718.

11. N. Levinson, *The first boundary value problem for* $\epsilon\Delta u + A(x, y)u_x + B(x, y)u_y + C(x, y)u = D(x, y)$ *for small* ϵ, Ann. of Math. (2) 51 (1950), 428–445. MR 11, 439.

12. A. Livne and Z. Schuss, *Singular perturbations for degenerate elliptic equations of second order*, Arch. Rational Mech. Anal. (to appear).

13. O. A. Oleinik, *On equations of elliptic type with a small parameter in the highest derivatives*, Mat. Sb. 31 (73) (1952), 104–117. (Russian) MR 14, 560.

14. S. V. Sivašinskiĭ, *The introduction of 'viscosity' into first order linear symmetric systems*, Vestnik Leningrad. Univ. 25 (1970), no. 19, 54–57. (Russian) MR 44 #593.

15. M. I. Višik and L. A. Ljusternik, *Regular degeneration and boundary layer for linear differential equations with a small parameter*, Uspehi Mat. Nauk 12 (1957), no. 5 (77), 3–122; English transl., Amer. Math. Soc. Transl. (2) 20 (1962), 239–364. MR 20 #2539; 25 #322.

DEPARTMENT OF MATHEMATICAL SCIENCES, TEL-AVIV UNIVERSITY, TEL-AVIV, ISRAEL